Even pairs in square-free Berge graphs with no odd prism

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Abstract

We consider the class $\mathcal{G}$ of Berge graphs that contain no odd prism and no square (cycle on four vertices). We prove that every graph G in $\mathcal{G}$ either is a clique or has an even pair, as conjectured by Everett and Reed. This result is used to devise a polynomial-time algorithm to color optimally every graph in $\mathcal{G}$.

Keywords: Berge graph, prism, square, even pair, coloring, algorithm

1 Introduction

A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$, where $\chi(H)$ is the chromatic number of H and $\omega(H)$ is the maximum clique size in H. In a graph G, a *hole* is a chordless cycle with at least four vertices and an *antihole* is the complement of a hole. Berge [1, 2, 3] introduced perfect graphs and conjectured that a graph is perfect if and only if it does not contain as an induced subgraph an odd hole or an odd antihole of length at least 5. A *Berge graph* is any graph that contains no odd hole and no odd antihole of length at least 5. This famous question (the Strong Perfect Graph Conjecture) was the objet of much research (see [14]), until it was proved by Chudnovsky, Robertson, Seymour and Thomas [6]: *Every Berge graph is perfect.* Moreover, Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [5] devised a polynomial-time algorithm that determines if a graph is Berge (hence perfect).

It is known that one can obtain an optimal coloring of a perfect graph in polynomial time due to the algorithm of Grötschel, Lovász and Schrijver [10]. This algorithm however is not purely combinatorial and impractical. Here are some ideas that could be fruitful in order to devise a purely combinatorial

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algorithm for coloring perfect graphs. An *even pair* in a graph G is a pair $\{x, y\}$ of non-adjacent vertices such that every chordless path between them has even length (number of edges). Given two vertices x, y in a graph G, the operation of *contracting* them means removing x and y and adding one vertex with edges to every vertex of $G \setminus \{x, y\}$ that is adjacent in G to at least one of x, y; we denote by G/xy the graph that results from this operation. Fonlupt and Uhry [9] proved that if G is a perfect graph and $\{x, y\}$ is an even pair in G, then the graph G/xy is perfect and $\chi(G/xy) = G$. In particular, given a $\chi(G/xy)$-coloring c of the vertices of G/xy, one can easily obtain a $\chi(G)$-coloring of the vertices of G as follows: keep the color for every vertex different from x, y; assign to x and y the color assigned by c to the contracted vertex. This idea could be the basis for a conceptually simple coloring algorithm for Berge graphs: as long as the graph has an even pair, contract any such pair; when there is no even pair find a coloring c of the contracted graph and, applying the procedure above repeatedly, derive from c a coloring of the original graph.

The algorithm for recognizing Berge graphs [5] can be used to detect an even pair in a Berge graph G; indeed, it is easy to see that two non-adjacent vertices a, b form an even pair in G if and only if the graph obtained by adding a vertex adjacent only to a and b is Berge. Thus, given a Berge graph G, one can try to color its vertices by keeping contracting even pairs until none can be found. Then some questions arise: which Berge graphs have no even pair, and which do not? What are the graphs for which a sequence of even-pair contractions leads to graphs that are easy to color?

Bertschi [4] proposed the following definitions. A graph G is *even-contractile* if either G is a clique or there exists a sequence $G_0, \dots, G_k$ of graphs such that $G = G_0$, for $i = 0, \dots, k-1$ the graph G_i has an even pair $\{x_i, y_i\}$ such that $G_{i+1} = G_i/x_iy_i$, and G_k is a clique. A graph G is *perfectly contractile* if every induced subgraph of G is even-contractile. This class is of interest because it turns out that many classical families of graphs are perfectly contractile; see [8].

Everett and Reed [8] proposed a conjecture aiming at a characterization of perfectly contractile graphs. A *prism* is a graph that consists of two vertex-disjoint triangles (cliques of size 3) with three vertex-disjoint paths P_1, P_2, P_3 between them, and with no other edge than those in the two triangles and in the three paths. The length of a path is its number of edges. Note that if two of the paths P_1, P_2, P_3 have lengths of different parities, then their union induces an odd hole. So in a Berge graph, the three paths of a prism have the same parity. A prism is *even* (resp. *odd*) if these three paths all have even lengths (resp. all have odd lengths).

Conjecture 1.1 ([8]). *A graph is perfectly contractile if and only if it contains no odd hole, no antihole of length at least* 5*, and no odd prism.*

Graphs that contain no odd hole, no antihole of length at least 5, and no odd prism were called *Grenoble graphs* by Bruce Reed.

The 'only if' part of Conjecture 1.1 is not hard to establish; see [11] for the details. The 'if' part of the conjecture remains open. A weaker conjecture was proposed by Everett and Reed [8] and eventually proved by Maffray and Trotignon [13], as follows.

Theorem 1.2 ([13])**.** *If a graph contains no odd hole, no antihole of length at least* 5*, and no prism then it is perfectly contractile.*

The proof of Theorem 1.2 is a polynomial time algorithm that takes as input any graph G that contains no odd hole, no antihole of length at least 5, and no prism, and produces a sequence of contractions of even pairs that turns G into a clique. Moreover, one can decide in polynomial time if a graph contains an odd hole, an antihole of length at least 5 or a prism [12].

A *square* is a hole of length four. A graph is *square-free* if it does not contain a square as an induced subgraph. Here we will study Conjecture 1.1 in square-free graphs. We will be able to prove that every square-free Grenoble graph that is not a clique has an even pair. Unfortunately, contracting an even pair may result in the presence of a square in the contracted graph (if the two vertices of the even pair were linked by a path of length four in the original graph). So it is difficult to establish that square-free Grenoble graphs are perfectly contractile. Nevertheless, using the presence of even pairs, we will prove the following theorem, which is the main result of this paper.

Theorem 1.3. *There exists a combinatorial and polynomial time algorithm which, given any square-free Grenoble graph G, returns an $\omega(G)$ coloring of G and a clique of size $\omega(G)$.*

Since Theorem 1.2 settles the case of graphs that have no prism, we may assume for our proof of Theorem 1.3 that we are dealing with a graph that contains an even prism. So the next sections focus on the study of such graphs. Note that results from [12] show that finding an induced prism in a Berge graph can be done in polynomial time.

We finish this section with some notation and terminology. In a graph G, given a set $T \subset V(G)$, a vertex of $V(G) \setminus T$ is *complete to T* if it is adjacent to all vertices of T. A vertex of $V(G) \setminus T$ is *anticomplete to T* if it is not adjacent to any vertex of T. Given two sets $S, T \subset V(G)$, S is *complete* to T if every vertex of S is complete to T, and S is *anticomplete* to T if every vertex of S is anticomplete to T. Given a path, any edge between two vertices that are not consecutive along the path is a *chord*. A path that has no chord is *chordless*.

2 Prisms

Several sections in the proof of the Strong Perfect Graph Theorem [6] are devoted to the analysis of Berge graphs that contain a prism. We extract here several theorems from [6] that we will use.

Let K be a prism, consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and three paths P_1, P_2, P_3, where each P_i has ends a_i and b_i, and for $1 \leq i < j \leq 3$ the only edges between $V(P_i)$ and $V(P_j)$ are a_ia_j and b_ib_j. The three paths P_1, P_2, P_3 are said to *form* the prism. Vertices a_1, a_2, a_3 and b_1, b_2, b_3 are the *corners* of the prism.

Theorem 2.1 ((7.3) in [6])**.** *In a Berge graph G, let R_1, R_2, R_3 be three chordless paths that form a prism K with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Assume that R_1, R_2, R_3 all have length at least 2. Let $Y \subset V(G)$ be anticonnected such that every vertex in Y is adjacent to at least two of a_1, a_2, a_3 and to at least two of b_1, b_2, b_3. Then at least two of a_1, a_2, a_3 and at least two of b_1, b_2, b_3 are complete to Y.*

Theorem 2.2 ((7.4) in [6])**.** *In a Berge graph G, let R_1, R_2, R_3 be three chordless paths that form a prism K with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Assume that R_1, R_2, R_3 all have length at least 2. Let R_1' be a chordless path from a_1' to b_1, such that R_1', R_2, R_3 also form a prism. Let $y \in V(G)$ have at least two neighbours in A and in B. Then y also has at least two neighbours in $\{a_1', a_2, a_3\}$.*

Theorem 2.3 ((10.1) in [6])**.** *In a Berge graph G, let R_1, R_2, R_3 be three chordless paths that form a prism K with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in K is not local. Assume no vertex in F is major with respect to K. Then there is a path f_1-…-f_n in F with $n \geq 1$, such that (up to symmetry) either:*

1. *f_1 has two adjacent neighbours in R_1, and f_n has two adjacent neighbours in R_2, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, and (therefore) G has an induced subgraph which is the line graph of a bipartite subdivision of K_4, or*

2. *$n \geq 2$, f_1 is adjacent to a_1, a_2, a_3, and f_n is adjacent to b_1, b_2, b_3, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, or*

3. *$n \geq 2$, f_1 is adjacent to a_1, a_2, and f_n is adjacent to b_1, b_2, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, or*

4. *f_1 is adjacent to a_1, a_2, and there is at least one edge between f_n and $V(R_3) \setminus \{a_3\}$, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K) \setminus \{a_3\}$.*

In this paper the above theorem will always be applied to graphs that do not contain any odd prism and (consequently) do not contain the line-graph of any bipartite subdivision of K_4. So only items 2, 3 or 4 hold. Moreover, it is not specified that the prism is even in the preceding theorem. We will use the following special case of this theorem.

Corollary 2.4. *In a Berge graph G, let R_1, R_2, R_3 be three chordless paths that form a prism K with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i and has even length. Let x be a vertex in $V(G)\setminus V(K)$ such that x is not a major neighbor of K and its set of attachments in K is not local. Then (up to symmetry) x is adjacent to a_1, a_2, and there is at least one edge between x and $V(R_3) \setminus \{a_3, b_3\}$, and there are no other edges between x and $V(K)\setminus\{a_3\}$. (In particular, x is anticomplete to $\{b_1, b_2, b_3\}$.)*

Theorem 2.5 ((10.3) in [6])**.** *Let G be a Berge graph, such that there is no nondegenerate appearance of K_4 in G. Let R_1, R_2, R_3 form a prism K in G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that no vertex in F is major with respect to K. Let x_1 be an attachment of F in the interior of R_1, and assume that there is another attachment x_2 of F not in R_1. Then there is a path f_1-…-f_n in F such that (up to the symmetry between A and B) f_1 is adjacent to a_2, a_3, and f_n has at least one neighbour in $R_1 \setminus a_1$, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K) \setminus \{a_1\}$.*

3 Hyperprisms

From now on, let G be a square-free Berge graph that contains an even prism.

We define hyperprisms as in [6]. Since G contains an even prism, $V(G)$ contains nine subsets

$$\begin{matrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{matrix}$$

with the following properties:

- These nine sets are nonempty and pairwise disjoint.
- For distinct $i, j \in \{1, 2, 3\}$, A_i is complete to A_j, and B_i is complete to B_j, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_j \cup B_j \cup C_j$.
- For each $i \in \{1, 2, 3\}$, every vertex of $A_i \cup B_i \cup C_i$ belongs to a chordless path between A_i and B_i with interior in C_i.

The 9-tuple $(A_1, C_1, B_1, A_2, C_2, B_2, A_3, C_3, B_3)$ is called a *hyperprism.* For each $i \in \{1, 2, 3\}$, a chordless path from A_i to B_i with interior in C_i is called an *i-rung.* Let us write $A = A_1 \cup A_2 \cup A_3$, $B = B_1 \cup B_2 \cup B_3$ and $C = C_1 \cup C_2 \cup C_3$. Let $S_i = A_i \cup B_i \cup C_i$ for $i \in \{1, 2, 3\}$. The triple (A_i, C_i, B_i) is called a *strip* of the hyperprism. We call (A, C, B) the *profile* of the hyperprism.

If we pick any i-rung R_i for each $i \in \{1, 2, 3\}$, we see that R_1, R_2, R_3 form a prism; any such prism is called an *instance* of the hyperprism. Since G contains no odd prism, every instance of the hyperprism is an even prism, and so every rung has even length.

Given two hyperprisms η and η' with profiles (A, C, B) and (A', C', B') respectively, we write $\eta \prec \eta'$ if $C \subseteq C'$ and either $A \subseteq A'$ and $B \subseteq B'$ or $A \subseteq B'$ and $B \subseteq A'$ and one of these inclusions is strict. Clearly, $\prec$ is an order relation on hyperprisms, so we can speak about maximal hyperprisms for $\prec$. Although the notion of profile does not appear in [6], it is easy to see that the notion of maximal hyperprism in [6] is equivalent to that which is defined here.

Let $\eta = (A_1, \dots, B_3)$ be a hyperprism, and let H be the subgraph of G induced on the union of these nine sets. A subset $X \subseteq V(H)$ is *local* (with respect to the hyperprism) if X is a subset of one of S_1, S_2, S_3, A or B. Let x be any vertex in $V(G) \setminus V(H)$. We say that x is a *major* neighbor of H is x is a major neighbor of an instance of H. Let M be the set of all major neighbors of H.

From now on, we assume that η is a maximal hyperprism.

Lemma 3.1. *For every connected subset F of $V(G) \setminus (V(H) \cup M)$, its set of attachments in H is local.*

This lemma is identical to Claim (2) in the proof of Theorem (10.6) in [6], so we omit its proof.

Lemma 3.2. *For each $i \in \{1, 2, 3\}$, $M \cup A_i \cup B_i$ is a cutset that separates C_i from $S_{i+1} \cup S_{i+2}$. Consequently, C_1, C_2 and C_3 lie in three distinct components of $G \setminus (M \cup A \cup B)$.*

Proof. For suppose on the contrary that there is a path $P = p\text{-}\cdots\text{-}q$, with $V(P) \subset V(G) \setminus (M \cup A_i \cup B_i)$ such that p has a neighbor in C_i and q has a neighbor in $S_{i+1} \cup S_{i+2}$. Let P be a shortest such path; then $V(P) \subseteq V(G) \setminus V(H)$, so P contradicts Lemma 3.1. □

Lemma 3.3. *Let $x \in M$. Then x is complete to at least two of A_1, A_2, A_3 and at least two of B_1, B_2, B_3.*

Proof. Since x is in M, there exists for each $i \in \{1, 2, 3\}$ an i-rung R_i such that x is a major neighbor of the prism K formed by R_1, R_2, R_3. Let R_i

have ends $a_i \in A_i$ and $b_i \in B_i$ $(i = 1, 2, 3)$. Consider any 1-rung P_1, and let K' be the prism formed by P_1, R_2, R_3. We claim that:

$$x \text{ is a major neighbor of } K'. \tag{1}$$

For suppose the contrary. Let X be the set of neighbors of x. Let P_1 have ends $a'_1 \in A_1$ and $b'_1 \in B_1$, and let $A' = \{a'_1, a_2, a_3\}$ and $B' = \{b'_1, b_2, b_3\}$. If $b'_1 = b_1$, then Theorem 2.2 shows that x has at least two neighbors in A', and so the claim holds. Therefore assume that $b'_1 \neq b_1$ and, similarly, that $a'_1 \neq a_1$. Let $\alpha = |X \cap A|$, $\beta = |X \cap B|$, $\alpha' = |X \cap A'|$, $\beta' = |X \cap B'|$. We know that $\alpha \geq 2$ and $\beta \geq 2$ since x is a major neighbor of K, and $\min\{\alpha', \beta'\} \leq 1$ since x is not a major neighbor of K'. Moreover, $\alpha' \geq \alpha - 1$ and $\beta' \geq \beta - 1$ since K and K' differ by only one rung. Up to the symmetry on A, B, these conditions imply that the vector $(\alpha, \beta, \alpha', \beta')$ is equal to either $(3, 2, 3, 1)$, $(3, 2, 2, 1)$, $(2, 2, 2, 1)$ or $(2, 2, 1, 1)$. In either case we have $\beta = 2$ and $\beta' = 1$, so x is adjacent to b_1, not adjacent to b'_1, and adjacent to exactly one of b_2, b_3, say to b_3.

Suppose that (α', β') is equal to $(3, 1)$ or $(2, 1)$. We can apply Theorem 2.3 to K' and $F = \{x\}$, and it follows that x satisfies item 4 of that theorem, so x is adjacent to a'_1, a_2, b_3 and has no neighbor in $V(K') \setminus (\{a'_1, a_2\} \cup V(R_3))$. In particular x has no neighbor in $V(R_2) \setminus \{a_2\}$, and then $V(R_2) \cup \{x, b_3\}$ induces an odd hole, a contradiction. So we may assume that $(\alpha, \beta, \alpha', \beta') = (2, 2, 1, 1)$, which restores the symmetry between A and B. Since $\alpha = 2$ and $\alpha' = 1$, x is adjacent to a_1, not adjacent to a'_1, and adjacent to exactly one of a_2, a_3. In fact if x is adjacent to a_2, then K' and $\{x\}$ violate Theorem 2.3. So x is adjacent to a_3 and not to a_2, and Theorem 2.3 implies that x is a local neighbor of K' with $X \cap K' \subseteq V(R_3)$, so x has no neighbor on P_1 or R_2.

We observe that for every 1-rung Q_1, the ends of Q_1 are either both adjacent to x or both not adjacent to x, for otherwise the prism formed by Q_1, R_2, R_3 and the set $F = \{x\}$ violate Theorem 2.3. Let $A'_1 = A_1 \setminus X$ and $A''_1 = A_1 \cap X$, and similarly $B'_1 = B_1 \setminus X$ and $B''_1 = B_1 \cap X$. The preceding observation means that every 1-rung is either between A'_1 and B'_1 or between A''_1 and B''_1. Let C'_1 be the set of vertices of C_1 that lie on a 1-rung whose ends are in $A'_1 \cup B'_1$, and let C''_1 be the set of vertices of C_1 that lie on a 1-rung whose ends are in $A''_1 \cup B''_1$. The sets C'_1 and C''_1 are disjoint and there is no edge between $A'_1 \cup C'_1 \cup B'_1$ and C''_1 or between $A''_1 \cup C''_1 \cup B''_1$ and C'_1, for otherwise we would find a 1-rung with one end in A'_1 and the other in B''_1. For every 1-rung P'_1 with ends in $A'_1 \cup B'_1$ Theorem 2.3 implies (just like for P_1) that x is a local neighbor of the prism formed by P'_1, R_2, R_3, so x has no neighbor on P'_1. Hence x has no neighbor in $A'_1 \cup C'_1 \cup B'_1$. We claim that A'_1 is complete to A''_1. For suppose on the contrary, up to relabelling vertices and rungs, that a'_1 and a_1 are not adjacent. Then $V(R_1) \cup \{x, a_1, a_2, b_3\}$ induces an odd hole. So the claim holds, and similarly B'_1 is complete to B''_1.

Now we consider S_2. Let $A'_2 = A_2 \setminus X$, $A''_2 = A_2 \cap X$, $B'_2 = B_2 \setminus X$ and

$B_2'' = B_2 \cap X$. By the same arguments as for the 1-rungs, we see that every 2-rung Q_2 is either between A_2' and B_2' or between A_2'' and B_2'', for otherwise the prism formed by P_1, Q_2, R_3 and the set $F = \{x\}$ violate Theorem 2.3. Let C_2' be the set of vertices of C_2 that lie on a 2-rung whose ends are in $A_2' \cup B_2'$, and let C_2'' be the set of vertices of C_2 that lie on a 1-rung whose ends are in $A_2'' \cup B_2''$. Then, by the same arguments as above, C_2' and C_2'' are disjoint and there is no edge between $A_2' \cup C_2' \cup B_2'$ and C_2'' or between $A_2'' \cup C_2'' \cup B_2''$ and C_2'. Also x has no neighbor in $A_2' \cup C_2' \cup B_2'$, and A_2' is complete to A_2'' and B_2' is complete to B_2''. It follows that the nine sets

$$\begin{array}{ccc} A_1' & C_1' & B_1' \\ A_2' & C_2' & B_2' \\ A_1'' \cup A_2'' \cup A_3 & C_1'' \cup C_2'' \cup C_3 \cup \{x\} & B_1'' \cup B_2'' \cup B_3 \end{array}$$

form a hyperprism, which contradicts the maximality of η. Thus (1) holds.

By (1) applied repeatedly, we obtain that x is a major neighbor of every instance of H.

Now suppose that x has a non-neighbor $u_1 \in A_1$ and a non-neighbor $u_2 \in A_2$. For each $i \in \{1, 2\}$ let P_i be an i-rung with end u_i, and let P_3 be any 3-rung. Then x is not a major neighbor of the prism formed by P_1, P_2, P_3, a contradiction. So x is complete to one of A_1, A_2, say to A_1. Likewise, x is complete to one of A_2, A_3. So x is complete to at least two of A_1, A_2, A_3. The same holds for B_1, B_2, B_3. This completes the proof of the lemma. $\square$

Lemma 3.4. *Let M be the set of major neighbors of η. Then:*
(i) Two of A_1, A_2, A_3 and two of B_1, B_2, B_3 are cliques.
(ii) M is complete to at least two of A_1, A_2, A_3 and at least two of B_1, B_2, B_3.
(iii) There is an integer $j \in \{1, 2, 3\}$ such that A_j and B_j are cliques and M is complete to $A_j \cup B_j$.

Proof. If (i) does not hold, then, up to symmetry, there are two non-adjacent vertices in A_1 and two non-adjacent vertices in A_2, and these four vertices induce a square, a contradiction.

(ii) We claim that M is complete to one of A_1, A_2. For suppose on the contrary that there are two non-adjacent vertices $a_1 \in A_1$ and $u \in M$ and also two non-adjacent vertices $a_2 \in A_2$ and $v \in M$. By Lemma 3.3, u is complete to A_2 and v is complete to A_1, so ua_2 and va_1 are edges, and $u \neq v$. If u and v are not adjacent, then, by Theorem 2.1 applied to K and $Y = \{u, v\}$, there is a vertex $b \in B$ that is complete to Y, and then $\{a_1, a_2, u, v, b\}$ induces a 5-hole, a contradiction. So u and v are adjacent, and $\{u, v, a_1, a_2\}$ induces a square, a contradiction. So the claim holds, say M is complete to A_1. Similarly, M is complete to one of A_2, A_3. Thus M is complete to two of A_1, A_2, A_3, and the same holds for B_1, B_2, B_3 by symmetry.

(iii) By (ii), we may assume that M is complete to $A_1 \cup B_1$. If both A_1, B_1 are cliques, then (iii) holds with $j = 1$. Therefore assume that A_1 is not a clique. By (i), A_2 and A_3 are cliques. Moreover M is complete to $A_2 \cup A_3$, for if there are non-adjacent vertices $u \in M$ and $a \in A_2 \cup A_3$, then by Lemma 3.3 the vertex u is complete to A_1, and then u, a and two non-adjacent vertices from A_1 induce a square. By (ii) M is complete to one of B_2, B_3, say to B_2. So if B_2 is a clique, then (iii) holds with $j = 2$. Therefore assume that B_2 is not a clique. Then B_3 is a clique by (i), moreover, as above (with A_1), M is complete to B_3. So (iii) holds with $j = 3$. Thus the lemma is proved. $\square$

3.1 Selecting a strip

Let us say that a strip (A_i, C_i, B_i) of the hyperprism is *good* if both A_i and B_i are cliques and M is complete to $A_i \cup B_i$. Lemma 3.4 says that every maximal hyperprism has a good strip. We may assume that (A_1, C_1, B_1) is a good strip of η. Moreover, we may assume that we choose η such that S_1 has the smallest size over all good strips of maximal hyperprisms.

Lemma 3.5. *Let $P = a$-u-$\cdots$-v-b be any chordless path with $a \in A_1$, $b \in B_1$, and $V(P) \cap M = \emptyset$. Then $V(P) \subset V(H)$. Moreover, either:*
- *P is a 1-rung, or*
- *the interior of P is an i-rung for some $i \in \{1, 2, 3\}$, or*
- *P has odd length, $V(P) \subseteq S_1$ and exactly one of $u \in A_1$ and $v \in B_1$ holds.*

Proof. Note that P has length at least 2. We prove the lemma by induction on the length of P. If P has length 2, say $P = a$-x-b, then we must have $x \in C_1$ (for otherwise, we could add x to C_1 and obtain a hyperprism that contradicts the maximality of η), and so P is a 1-rung. Now assume that the length of P is at least 3. Let $\tilde{P}$ be the interior of P.

When $V(P) \not\subset V(H)$, there are subpaths $P_1, \ldots, P_k$ of P, with k odd, $k \geq 3$, such that $P = P_1$-P_2-$\cdots$-P_k, with $a \in V(P_1)$ and $b \in V(P_k)$, and, for all odd j, $V(P_j) \subset V(H)$, and for all even j, $V(P_j) \cap V(H) = \emptyset$. When $V(P) \subset V(H)$ we use the same notation, with $k = 1$. When $k \geq 3$, for each even j, let X_j be the set of attachment of P_j in H. We claim that:

$$\text{For each even } j\text{, there is } i_j \in \{1, 2, 3\} \text{ such that } X_j \subseteq S_{i_j}. \tag{2}$$

By Lemma 3.1 applied to P_j, we know that X_j is local with respect to H. Suppose that $X_j \subseteq A$. Let w (resp. w') be the vertex in P_{j-1} (resp. in P_{j+1}) that has a neighbor in P_j. Then $w, w' \in X_j$, and w, w' are not adjacent, so $w, w' \in A_2 \cup A_3$ and a is adjacent to both w, w', a contradiction. Hence X_j is not a subset of A and, similarly, not of B either. Thus (2) holds.

Suppose that $u \in A_2$. Then $V(P_1) \cap A = \{a, u\}$ (for otherwise a would have a neighbor on $P \setminus \{a, u\}$), and so $V(P_1 \setminus a) \subseteq S_2$. Now, applying (2)

repeatedly, we obtain that for each even j we have $X_j \subseteq S_2$, for each odd j with $j < k$ we have $V(P_j) \subseteq S_2$, and $V(P_k \setminus b) \subseteq S_2$. Then $V(\tilde{P}) \subseteq S_2$, for otherwise we could add the vertices of $\tilde{P}$ to S_2 and thus obtain a hyperprism that contradicts the maximality of η. Hence $\tilde{P}$ is a 2-rung and the lemma holds. We obtain a similar conclusion if either $u \in A_3$ or $v \in B_2 \cup B_3$. Now assume that $u \notin A_2 \cup A_3$ and $v \notin B_2 \cup B_3$.

Suppose that $u \notin A_1$ and $v \notin B_1$. Then $V(P_1) \cap A = \{a\}$ and $V(P_1) \subseteq S_1$. Now, applying (2) repeatedly, we obtain that for each even j we have $X_j \subseteq S_1$, for each odd j with $j < k$ we have $V(P_j) \subseteq S_1$, and $V(P_k \setminus b) \subseteq S_1$. Then $V(P) \subseteq S_1$, for otherwise we could add the vertices of P to S_1 and thus obtain a hyperprism that contradicts the maximality of η. Hence P is a 1-rung and the lemma holds.

Now suppose that $u \in A_1$ and $v \in B_1$. We can apply induction to $\tilde{P}$. It cannot be that the second or third item of the lemma holds for $\tilde{P}$ (for otherwise a would have two neighbors on P), so the first item holds for $\tilde{P}$, and so the second item holds for P.

Finally suppose, up to symmetry, that $u \in A_1$ and $v \notin B_1$. We can apply induction to $P \setminus a$. It cannot be that the second or third item of the lemma holds for $P \setminus a$, so the first item holds for $P \setminus a$, and so the third item holds for P. Thus the lemma holds. $\square$

A *necklace* is a graph that consists of four disjoint chordless paths $R_1 = a \cdots a'$, $R_2 = b \cdots b'$, $R_3 = c \cdots c'$, $R_4 = d \cdots d'$, where R_1, R_2 may have length 0 but R_3, R_4 have length at least 1, and such that the edge-set of S is $E(R_1) \cup E(R_2) \cup E(R_3) \cup E(R_4) \cup \{a'c, a'd, cd, b'c', b'd', c'd'\}$. Note that $\{a', c, d\}$ and $\{b', c', d'\}$ are triangles in S. Vertices a and b are the endvertices of the necklace, and we may also say that S is an (a, b)-necklace.

Let R' and R'' be two 1-rungs, where R' has ends u', w, and R'' has ends u'', w, and $u' \neq u''$ (so w is in one of the two sets A_1, B_1 and u', u'' are in the other set). We say that R' and R'' *converge* if u' has no neighbor in $R'' \setminus u''$ and u'' has no neighbor in $R' \setminus u'$.

Lemma 3.6. *There do not exist two* 1*-rungs that converge.*

Proof. Suppose on the contrary that R' and R'' are two 1-rungs that converge. Choose R' and R'' such that $|V(R') \cup V(R'')|$ is minimized. Let $R' = u_0\text{-}u_1\text{-}\cdots\text{-}u_p$ (with p even, $p \geq 2$) and $R'' = v_0\text{-}v_1\text{-}\cdots\text{-}v_q$ (with q even, $q \geq 2$), and assume up to symmetry that $u_0, v_0 \in A_1$, $u_0 \neq v_0$, $u_p = v_q \in B_1$, u_0 has no neighbor in $R'' \setminus v_0$, and v_0 has no neighbor in $R' \setminus u_0$. Let i be the smallest integer such that u_i has a neighbor in $R'' \setminus v_0$. Note that i exists since u_{p-1} has a neighbor in $R'' \setminus v_0$. Also $i \neq 0$ because of the hypothesis on u_0. Likewise, let j be the smallest integer such that v_j has a neighbor in $R' \setminus u_0$. Let h be the smallest integer such that $u_i v_h$ is an edge. So $0 < j \leq h$. Moreover, $h < q$, for otherwise we must have $i = p - 1$ and $V(R') \cup V(R'')$

induces an odd hole. Now the set $\{u_0, \dots, u_i, v_0, \dots, v_h\}$ induces a hole, so it is an even hole, so i and h have the same parity. We claim that:

$$\text{We may assume that } R'[u_{i+1}, u_p] = R''[v_{j+1}, v_q]. \tag{3}$$

To prove this, first suppose that $i \neq p-1$. Let k be the largest integer such that u_iv_k is an edge. Then u_0-u_1-$\cdots$-u_i-v_k-$\cdots$-v_q is a chordless path, so it is a 1-rung, and it must have even length, so h and k have different parities. If $k \neq h+1$, then v_0-v_1-$\cdots$-v_h-u_i-v_k-$\cdots$-v_q is a chordless path, so it is a 1-rung, and it has odd length, a contradiction. Hence $k = h+1$. The minimality of $|V(R') \cup V(R'')|$ implies that $R'[u_{i+1}, u_p] = R''[v_{h+1}, v_q]$, so $h = j$ and the claim holds. Therefore we may assume that $i = p-1$. By the same argument as with i, we may assume that $j = q-1$ (so $h = q-1$). Thus (3) holds.

Let R_2 be any 2-rung, with ends $a_2 \in A_2$ and $b_2 \in B_2$. Let $P_1 = u_0$-u_1-$\cdots$-u_i, $P_2 = v_0$-v_1-$\cdots$-v_j, and $P_3 = a_2$-R_2-b_2-u_p-u_{p-1}-$\cdots$-u_{i+1}. It follows from (3) that P_1, P_2, P_3 form a prism. Since G contains no odd prism, these three paths have even length, and so i and j are even, so $u_{i+1} \neq u_p$ and $u_{i+1} \in C_1$. Let $Z = C_2 \cup C_3 \cup B_2 \cup B_3 \cup \{u_{i+2}, \dots, u_p\}$. We observe that:

$$\begin{array}{ccc}
\{u_0\} & \{u_1, \dots, u_{i-1}\} & \{u_i\} \\
\{v_0\} & \{v_1, \dots, v_{i-1}\} & \{v_j\} \\
A_2 \cup A_3 & Z & \{u_{i+1}\}
\end{array}$$

form a hyperprism η'. So there exists a maximal hyperprism η^* such that $\eta' \preceq \eta^*$. Let $\eta^* = (A_1^*, C_1^*, B_1^*, A_2^*, C_2^*, B_2^*, A_3^*, C_3^*, B_3^*)$, $A^* = A_1^* \cup A_2^* \cup A_3^*$, $B^* = B_1^* \cup B_2^* \cup B_3^*$ and $C^* = C_1^* \cup C_2^* \cup C_3^*$, and, for each $i \in \{1,2,3\}$, $S_i^* = A_i^* \cup C_i^* \cup B_i^*$. We know that $\{u_0, v_0\} \cup A_2 \cup A_3 \subseteq A^*$, and $\{u_i, v_j, u_{i+1}\} \subseteq B^*$, and $\{u_1, \dots, u_{i-1}\} \cup \{v_1, \dots, v_{i-1}\} \cup Z \subseteq C^*$. Since Z is connected, we may assume, up to symmetry, that $Z \subseteq C_3^*$, and so $A_2 \cup A_3 \subseteq A_3^*$ and $\{u_{i+1}\} \subseteq B_3^*$. We claim that:

$$S_1^* \cup S_2^* \subset S_1, \text{ and } A_1^* \cup A_2^* \subset A_1 \text{ and } B_1^* \cup B_2^* \cup B_3^* \subset C_1. \tag{4}$$

We may assume up to symmetry that P_2 is either a 2-rung or a 3-rung of η^*. Let R_1^* be any 1-rung of η^*, with ends $a_1^* \in A_1^*$ and $b_1^* \in B_1^*$. So a_1^* is complete to $A_2 \cup A_3 \cup \{v_0\}$ and b_1^* is complete to $\{v_j, u_{i+1}\}$, and there are no other edges between $V(R_1^*)$ and $V(P_2) \cup V(P_3)$. Let $R_1 = a_1^*$-R_1^*-b_1^*-u_{i+1}-R'-u_p; so R_1 is an even chordless path. Let $R_1^+ = v_0$-a_1^*-R_1-u_p; so R_1^+ is an odd chordless path. By Lemma 3.5, we have $a_1^* \in A_1$ and R_1 is a 1-rung of η. Thus $V(R_1^*) \subset A_1 \cup C_1$ for every 1-rung R_1^* of η^*, and $A_1^* \subset A_1$ and $B_1^* \subset C_1$. We see that R_1 converges with R'', so we may let R_1^* play the role of R', which restores the symmetry between 1-rungs and 2-rungs of η^*, and consequently $V(R_2^*) \subset S_2$ holds for every 2-rung R_2^* of η^*. Thus (4) holds.

Let M^* be the set of major neighbors of η^*. We claim that:

$$M^* \text{ is complete to } A_1^* \cup A_2^*. \tag{5}$$

For suppose that some vertex $m^* \in M^*$ is not is complete to $A_1^* \cup A_2^*$. Then m^* is complete to A_3^* and in particular to $A_2 \cup A_3$. Moreover $m^* \notin A_1$, since A_1 is a clique and $A_1^* \cup A_2^* \subset A_1$. Therefore $m^* \notin V(H)$. We know that m^* is complete to one of B_1^*, B_2^*, which are subsets of C_1. Hence the set of attachments of m^* in H is not local, so Lemma 3.1 implies that $m^* \in M$, so m^* is complete to A_1, a contradiction. Thus (5) holds.

$$\text{For some } j \in \{1, 2\},\ (A_j^*, C_j^*, B_j^*) \text{ is a good strip of } \eta^*. \tag{6}$$

Since $A_1^* \cup A_2^* \subseteq A_1$, both A_1^* and A_2^* are cliques. We may assume up to symmetry that M^* is complete to B_1^*. So if B_1^* is a clique, the claim holds with $j = 1$. Now assume that B_1^* is not a clique. Then B_2^* is a clique by Lemma 3.4 applied to η^*, and M^* is complete to B_2^* (for otherwise two non-adjacent vertices from $M^* \cup B_2^*$ plus two non-adjacent vertices from B_1^* induce a square, and so the claim holds with $j = 2$. Thus (6) holds.

Now Claims (4) and (6) contradict the choice of η (with the smallest good strip). This completes the proof of the lemma. □

3.2 Finding an even pair

Pick any $b \in B_1$. For any two $a, a' \in A_1$, write $a <_b a'$ whenever there exists an odd chordless path R from a to b such that a' is the neighbor of a on R. Note that in that case, Lemma 3.5 implies that $R \setminus a$ is a 1-rung.

Lemma 3.7. *For each $b \in B_1$, $<_b$ is an order relation.*

Proof. We first claim that the relation $<_b$ is antisymmetric. Suppose on the contrary that there are vertices $u, v \in A_1$ such that $u <_b v$ and $v <_b u$. So there exists an odd chordless path $P_u = u\text{-}v\text{-}\cdots\text{-}b$ and there exists an odd chordless path $P_v = v\text{-}u\text{-}\cdots\text{-}b$. By Lemma 3.5, $P_u \setminus u$ and $P_v \setminus v$ are 1-rungs. Because of b these two rungs converge, which contradicts Lemma 3.6. So $<_b$ is antisymmetric. Now we claim that $<_b$ is transitive. Let u, v, w be three vertices in A_1 such that $u <_b v <_b w$. So there is an odd chordless path $v\text{-}w_0\text{-}w_1\text{-}\cdots\text{-}w_k$ with k even, $k \geq 2$, $w = w_0$ and $w_k = b$. By Lemma 3.5, $w_0\text{-}w_1\text{-}\cdots\text{-}w_k$ is a 1-rung. Let j be the largest integer such that uw_j is an edge. Suppose that $j > 0$. If j is even, then $u\text{-}w_j\text{-}\cdots\text{-}w_k$ is a 1-rung of odd length, a contradiction. If j is odd, then $v\text{-}u\text{-}w_j\text{-}\cdots\text{-}w_k$ is an odd chordless path, so $v <_b u$, which contradicts the fact that $<_b$ is antisymmetric. So $j = 0$, which implies that $u <_b w$. Hence $<_b$ is antisymmetric and transitive, so it is an order relation. □

Similarly, for each $a \in A_1$, and for any two $b, b' \in B_1$, we write $b <_a b'$ whenever there exists an odd chordless path R from b to a such that b' is the neighbor of b on R. So $<_a$ is an order relation on B_1 for each a.

Lemma 3.8. *If there are four vertices $a, u \in A_1$ and $b, v \in B_1$ such that $a <_b u$ and $b <_u v$, then $a <_v u$.*

Proof. The hypothesis that $a <_b u$ means that there is an odd chordless path $R = a\text{-}r_0\text{-}r_1\text{-}\cdots\text{-}r_k$ with $r_0 = u$ and $r_k = b$. By Lemma 3.5, $R \setminus a$ is a 1-rung, so k is even. The hypothesis that $b <_u v$ means that there is an odd chordless path $Q = b\text{-}v\text{-}\cdots\text{-}u$, and, by Lemma 3.5, $Q \setminus b$ is a 1-rung. If v has no neighbor in $R \setminus b$, then $R \setminus a$ and $Q \setminus b$ are two rungs that converge, a contradiction. So there is an integer $j < k$ such that vr_j is an edge, and we choose the smallest such j. So $r_0\text{-}r_1\text{-}\cdots\text{-}r_j\text{-}v$ is a 1-rung, so j is odd. Then $a\text{-}r_0\text{-}r_1\text{-}\cdots\text{-}r_j\text{-}v$ is an odd chorldess path, which shows that $a <_v r_0$, i.e., $a <_v u$. □

Lemma 3.9. *There exists an even pair $\{a, b\}$ with $a \in A_1$ and $b \in B_1$.*

Proof. For each $a \in A_1$, let $\mathrm{Max}(a)$ be the set of maximal elements of the partially ordered set $(B_1, <_a)$. Likewise, for each $b \in B_1$, let $\mathrm{Max}(b)$ be the set of maximal elements of $(A_1, <_b)$. We claim that:

$$\text{There exist } a \in A_1 \text{ and } b \in B_1 \text{ such that } a \in \mathrm{Max}(b) \text{ and } b \in \mathrm{Max}(a). \quad (7)$$

For each $a \in A_1$ and $b \in B_1$, let $D(a, b) = \{a' \in A \mid a' <_b a\}$. Choose a and b such that the size of $D(a, b)$ is maximized. We have $a \in \mathrm{Max}(b)$, for otherwise, there is $u \in A_1$ such that $a <_b u$, so $D(u, b) \supseteq D(a, b) \cup \{a\}$, which contradicts the choice of a and b. So if $b \in \mathrm{Max}(a)$ the claim holds. Hence let us assume that $b \notin \mathrm{Max}(a)$. This means that there exists $v \in \mathrm{Max}(a)$ such that $b <_a v$. If $a \in \mathrm{Max}(v)$, then the claim holds with the pair a, v. Hence let us assume that $a \notin \mathrm{Max}(v)$. So there exists $u \in \mathrm{Max}(v)$ such that $a <_v u$. For each $a' \in D(a, b)$, we can apply Lemma 3.8 to the four vertices a', a, b, v, which implies $a' <_v a$ and (by the transitivity of $<_v$) $a' <_v u$. So $D(u, v) \supseteq D(a, b) \cup \{a\}$, which contradicts the choice of a and b. Thus (7) holds.

Let a, b be any two vertices that satisfy (7). We claim that $\{a, b\}$ is an even pair of G. For suppose that there exists an odd chordless path P with ends a and b. By Lemma 3.5, and up to symmetry, we may assume that the neighbor a' of a on P is in A_1 and that $P \setminus b$ contains no vertex of B_1. This means that $a <_b a'$, which contradicts the fact that $a \in \mathrm{Max}(b)$. So the lemma holds. □

Let $A_1 = \{a_1, \ldots, a_k\}$ and $B_1 = \{b_1, \ldots, b_\ell\}$, and assume up to symmetry that $k \leq \ell$. By Lemma 3.9, we may assume up to relabeling that $\{a_1, b_1\}$ is an even pair of G. Similarly, for $i = 2, \ldots, k$, we may assume that $\{a_i, b_i\}$ is an even pair of $G \setminus \{a_1, b_1, \ldots, a_{i-1}, b_{i-1}\}$.

3.3 Decomposing the graph

By Lemma 3.2, the set $M \cup A_1 \cup B_1$ is a cutset of G, so $V(G) \setminus (M \cup A_1 \cup B_1)$ can be partitioned into two subsets X and Y, with $C_1 \subseteq X$ and $C_2 \subset Y$, such that there is no edge between X and Y. Let $G_X = G \setminus Y$ and $G_Y = G \setminus X$. Thus we consider that G is decomposed into G_X and G_Y. Since G_X and G_Y are proper induced sugraphs of G, we may assume by induction that we have a clique Q_X of G_X of size $\omega(G_X)$ and a coloring c_X of G_X with $\omega(G_X)$ colors, and the same for G_Y.

Lemma 3.10. *There exists a coloring c'_X of G_X with $\omega(G_X)$ colors such that $c'_X(a_i) = c'_X(b_i)$ for all $i = 1, \dots, k$, and such a coloring can be obtained from c_X in polynomial time.*

Proof. Suppose that c_X itself does not have the property described in the lemma, and let h be the smallest integer such that $c_X(a_h) \neq c_X(b_h)$. In case $h > 1$, we may assume, up to relabeling, that $c_X(a_i) = i = c_X(b_i)$ for all $i = 1, \dots, h-1$. Let $c_X(a_h) = i$ and $c_X(b_h) = j$, with $i \neq j$. Note that both $i, j > h-1$. Let $H_{i,j}$ be the bipartite subgraph of G induced by the vertices of color i and j. We swap colors i and j in the component of H that contains a_h. This component does not contain b_h, for otherwise it contains a chordless odd path between a_h and b_h, and this path is in $G \setminus \{a_1, b_1, \dots, a_{h-1}, b_{h-1}\}$ since it contains no vertex of color less than i and j; but this contradicts the fact that $\{a_h, b_h\}$ is an even pair of $G \setminus \{a_1, b_1, \dots, a_{h-1}, b_{h-1}\}$. So after this swapping vertices a_h and b_h have the same color. Thus we obtain a coloring of G_X with $\omega(G_X)$ colors where the value of h has increased. Repeating this procedure at most k times leads to the desired coloring. □

Applying Lemma 3.10 to both G_X and G_Y, we obtain colorings c_X and c_Y of G_X and G_Y respectively such that, up to relabeling, $c_X(a_i) = c_X(b_i) = c_Y(a_i) = c_Y(b_i) = i$ for each $i = 1, \dots, k$. Recall that $M \cup (B \setminus \{b_1, \dots, b_k\})$ is a clique and that all its vertices are adjacent to at least one of a_i, b_i for each $i = 1, \dots, k$. So we may assume, up to relabeling, that every vertex z in $M \cup (B \setminus \{b_1, \dots, b_k\})$ satisfies $c_X(z) = c_Y(z)$ too. It follows that the two colorings c_X and c_Y can be merged into a coloring of G. This coloring uses $\max\{\omega(G_X), \omega(G_Y)\}$ colors, and one of Q_X and Q_Y is a clique of that size. So the coloring and the larger of these two cliques are both optimal.

3.4 The algorithm

We can now describe our algorithm.

Input: A graph G on n vertices.
Output: Either a coloring of G and a clique of the same size, or the answer "G is not a square-free Grenoble graph".

Procedure:
1. First test whether G is square-free, and test whether G is Berge with the algorithm from [5]. Then test whether G contains a prism as explained in [12]. If these tests produce an induced subgraph of G that is either a square, or an odd hole, or an odd prism, return the answer "G is not a square-free Grenoble graph" and stop. If the algorithm from [12] shows that G contains no prism, then color G applying the algorithm from [13].
2. Now suppose that G contains an even prism. Grow a maximal hyperprism η, and find a good strip S_1 of η.
Apply the proof of Lemma 3.7 to every vertex $x \in A_1 \cup B_1$. That proof either establishes that $<_x$ is an order relation or finds 1-rungs that converge; in the latter case, apply the proof of Lemma 3.6 to obtain a new maximal hyperprism with a smaller good strip, and restart from that hyperprism.
When $<_x$ is an order relation for all $x \in A_1 \cup B_1$, Lemma 3.9 shows how to find even pairs. The graph G is decomposed into graphs G_X and G_Y, and an optimal coloring and a maximal clique for G can be obtained as explained above.

Let us analyse the complexity of the algorithm. One can decide whether a given graph G is Berge in time $O(n^9)$ with the algorithm from [5]. One can test whether G is square-free in time $O(n^4)$, and whether a Berge graph G contains a prism in time $O(n^5)$ as explained in [12]. Now assume that the algorithm produces an even prism. It is easy to to see that all the procedures in part 2 of algorithm (growing a maximal hyperprism, determining the orderings) can be performed in time at most $O(n^3)$, and we make additional remarks. First remark that when we need to restart from a new hyperprism, the size of the good strip is strictly smaller, and so this restarting step occurs at most $O(n)$ times. Secondly, remark that when G is decomposed into graphs G_X and G_Y, the algorithm is called recursively on them. This defines a decomposition tree T for G: every decomposition node of T is an induced subgraph G' of G and has two children which are induced subgraphs of G'; and every leaf of T is a graph that contains no prism. Let us show that this tree has polynomial size. When G is decomposed into graphs G_X and G_Y as above, because of a certain cutset that arises from a hyperprism η, we mark the corresponding node of the tree with a pair of vertices $\{c_1, c_2\}$ where $c_1 \in C_1$ and $c_2 \in C_2$ are chosen arbitrarily. We mark every subsequent decomposition node similarly. Note that only pairs of non-adjacent vertices are used to mark any node.

Lemma 3.11. *Every pair of vertices of G is used to mark at most one node of the decomposition tree.*

Proof. Without loss of generality let us consider the node G itself, decomposed into graphs G_X and G_Y along a cutset $M \cup A_1 \cup B_1$ corresponding to a hyperprism η, with the same notation as above. Let T_X be the subtree of T whose root is G_X, and define T_Y similarly. The node G of T is marked with a pair of vertices $\{c_1, c_2\}$ where $c_1 \in C_1$ and $c_2 \in C_2$. Since $c_1 \notin Y$ and $c_2 \notin X$, the pair $\{c_1, c_2\}$ is not included in the vertex-set of any descendant of G in the tree; so this pair will not be used to mark any node of T other than G.

Now suppose that a pair $\{c, d\}$ is used to mark a node in T_X and also a node in T_Y. Then $\{c, d\} \subseteq V(G_X) \cap V(G_Y) = M \cup A_1 \cup B_1$, and since c and d are not adjacent, we have $c \in A_1$ and $d \in B_1$. Since $\{c, d\}$ marks a node in T_X, there is a hyperprism η_X in G_X such that c and d lie in the interior of two distinct strips of η_X. Let R_c and R_d be rungs of η_X that contain c and d respectively (so R_c and R_d lie in different strips of η_X), and let R be a rung in the third strip of η_X. Let K be the prism formed by R_c, R_d, R. So $V(K) \subseteq V(G_X)$. Since $c \in A_1$, and A_1 is a clique, and c lies in the interior of R_c, it follows that A_1 contains at most one corner of K. Likewise B_1 contains at most one corner of K. This implies that the set of major neighbors of K is included in G_X. Moreover, if A_1 contains a corner u of K and B_1 contains a corner v of K, then u and v are not in the same rung of K (for otherwise c and d would also lie on that same rung). Let R_2 be any 2-rung in η. Then R_2 contains no major neighbor of K, and R_2 satisfies the hypothesis of Theorem 2.3 with respect to K. The preceding observations imply that R_2 must satisfy item 1 of Theorem 2.3, and consequently G contains an odd prism, a contradiction. So one of T_X, T_Y is such that none of its nodes is marked with $\{c, d\}$. (Actually the preceding argument holds for T_Y as well, so any pair $\{c, d\}$ with $c \in A_1$ and $d \in B_1$ will never be used to mark any node of T.)

The preceding two paragraphs, repeated for every node of T, imply the validity of the lemma. $\square$

By Lemma 3.11 the total number of nodes in T is $O(n^2)$. The leaves of the decomposition tree T are Berge graphs with no antihole (since they are square-free) and no prism, so they can be colored in time $O(n^6)$ as explained in [13]. At each node G' of T different from the root G, we know that G' is an induced subgraph of G, so it is square-free Berge; hence we must only test whether G' contains a prism, which is done in time $O(n^5)$ as explained in [12]. So the total complexity of the algorithm is $O(n^9 + n^2 \times n^5 + n^2 \times n^6) = O(n^9)$. This completes the proof of Theorem 1.3.

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